MTH 132 Exam 3 Topics

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Using derivatives to sketch curves You should know what the derivative of f tells you about the original function (and in fact, this is the theme of the entirety of Chapter 4). In particular, we have

- If $f'(x) > 0$ on an interval, the function is increasing there.
- If $f'(x) < 0$ on an interval, the function is decreasing there.

Make sure you can use the pictures here to come up with the first derivative test: If a function changes from increasing to decreasing, it must do so through a peak - so the function has to have a maximum there. Likewise, you should know what the second derivative tells you in terms of concavity: Positive second derivative means increasing rate of change means concave up, and vice versa.

Know the definition and how to find critical points: These are where f' is zero, or undefined. These are useful because they give us a list of possible maxima and minima for a function. So for example, if we want to determine where $x^3 - x$ has local extrema, we can take the derivative to get

$$3x^2 - 1 = 0 \implies x = \pm\frac{1}{\sqrt{3}}$$

Now taking the second derivative, we get $6x$, which is positive at $1/\sqrt{3}$, and negative at $-1/\sqrt{3}$. From the second derivative test, this tells us that $1/\sqrt{3}$ must have a local minimum, and the other critical point is a local maximum.

Remember that we also need to check endpoints: Even if a function has critical points in the interior, these don't have to give us **global** maxima or minima (nor even be extreme points at all!). For example, the function x^3 on the interval $[-1, 2]$ has its minimum at -1, maximum at 2, but nothing at all happens at the critical point $x = 0$.

Now to put this all together to come up with curve sketches, interpret these geometrically. Make sure you can picture things like 'decreasing, concave up' and so on. Generally, you can fit the rough picture together with the x- and y-intercepts of the function (and any asymptotes, if they exist) to give a sketch of the graph. See the quiz 4 solutions for an example.

Using derivatives to optimize things From the previous part, one of the most important applications of derivatives is to help us find where a function is maximized or minimized. This gives us a setup for the following problem:

Maximize (minimize) the function $f(x)$ on the interval I.

The technique is this:

1. Find appropriate bounds for the input, from the underlying physical problem (e.g. no negative lengths are allowed).

2. Find the critical points, and evaluate f there. Use a derivative test to classify as maxima / minima as needed.

3. Evaluate f at the endpoints. List the results together with the result from the previous step. The biggest one is the maximum, the smallest one is the minimum.

So let's do an example: Suppose we want to build a rectangular fence (with length l and width w) with area 100; we want to use as little fence as possible. We can write the amount of fence required as

$$2l + 2w$$

since that's the perimeter of the region. Now we also have that $lw = 100$, since that's the area. Using this to eliminate one variable, we find

$$l = \frac{100}{w}$$

so that we want to minimize the function

$$P(w) = \frac{200}{w} + 2w$$

The domain here is $w > 0$: we can't have negative (or zero) length, but we could have a *very* long and skinny fence. To find the critical points,

$$P'(w) = -\frac{200}{w^2} + 2 = 0 \implies w^2 = 100 \implies w = 10$$

We check that this is actually a minimizer:

$$P''(w) = \frac{400}{w^3} \implies P''(10) = 0.4 > 0$$

so by the second derivative test, this works. Notice that this gives us a square region - so the most efficient way to use the fence is to build a square.

Antiderivatives You should know how to find the antiderivatives of simple functions; there are a couple ways to do this:

- Recognize the function as the antiderivative of something familiar; for example,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

  For derivatives, we bring the power down and subtract 1; here, we do the exact opposite by adding one and putting the new power in the denominator. The other two main functions we know are $\int \sin x dx = -\cos x + C$ and $\int \cos x = \sin x + C$.

- Do some algebraic simplification to turn it into something you're familiar with. If you want to integrate

$$(2t^3 + 1)^2$$

  then start by expanding it as $4t^6 + 4t^3 + 1$; this only has power functions, which we can integrate easily.

- Guess the correct form, if the function only differs from something known by a constant. So if we want to integrate
$$(3x+1)^{100}$$
we might start by guessing (just like with pure power functions) that the antiderivative is
$$(3x+1)^{101}$$
Differentiating, we find that
$$\frac{d}{dx}(3x+1)^{101} = 101(3x+1)^{100}\cdot 3$$
per the chain rule, so we're off by a factor of $3 \cdot 101 = 303$. We divide to fix this, getting
$$\int (3x+1)^{100}dx = \frac{(3x+1)^{101}}{303} + C$$

You should also know some of the basic properties of indefinite integrals, such as
$$\int f(x)+g(x)dx = \int f(x)dx + \int g(x)dx$$

Other things You should know how to compute definite integrals by interpreting them as areas of geometric objects (e.g. circles, rectangles, triangles, etc.). You should also know how to combine definite integrals via sum / difference rules, and how to combine the limits of integration.

You should know how to use sigma notation; for example, make sure that you can recognize patterns in terms of a changing index. For example, to express
$$3+4+5+6+\cdots+50$$
we can recognize this as an index starting at 3 and ending at 50, changing by 1 at each step. So one way to write the sum is
$$\sum_{k=3}^{50} k$$

You should know the statement of Rolle's theorem and the Mean Value theorem, and how to apply them / verify their conclusions. For example, the function $g(x) = x^3 - x$ has $g(0) = g(1)$, so Rolle's theorem implies that the derivative is zero somewhere between 0 and 1. By a direct computation, we find that $g'(1/\sqrt{3}) = 0$, verifying the conclusion of Rolle's theorem.

This isn't a complete list of topics, or what can be covered on the exam, but it's a good place to start. Make sure that you can draw the pictures for these concepts - don't memorize the derivative tests, but rather understand them via the pictures. There are many problems in the textbook (from the Chapter 4 review), as well as WeBWorK that you can use to review; any material from chapter 4 or sections $5.1 - 5.3$ (with the exception of Newton's method) is fair for the exam.